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2007 J. Phys. A: Math. Theor. 40 2219

(http://iopscience.iop.org/1751-8121/40/9/023)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 2219–2226 doi:10.1088/1751-8113/40/9/023

A rigorous framework for Leslie's model of homogeneous anisotropic turbulence

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Received 30 June 2006, in final form 12 December 2006
Published 14 February 2007
Online at stacks.iop.org/JPhysA/40/2219

Abstract
Leslie, in the book *Developments in the Theory of Turbulence* (1973 Oxford: Clarendon), offered a very simple and intuitively founded model for the case of homogeneous anisotropic turbulence. Here, we offer a rigorous reformulation of Leslie's model leading to a general form for the velocity correlation tensor that satisfies realizability conditions like symmetry in its tensor indices and the condition of solenoidality arising from the incompressibility condition. The anisotropic part of the correlation tensor involves two non-dimensional constants—one arising from the pure strain term and the other from the term that induces distortion in the wave vector space; the rapid pressure term does not contribute. The estimates for these non-dimensional constants yield encouraging results within this framework.

PACS numbers: 47.27.Gs, 47.27.Jv

1. Introduction

Most turbulent flows occurring in Nature are highly anisotropic and thus the understanding of anisotropic turbulence is not only of practical importance but is also a theoretical challenge. The well-known Kolmogorov phenomenology [1–4] assumes isotropy for the small-scale turbulent motions—the effect of any anisotropy introduced at the large scales is expected to diminish rapidly due to successive loss of memory as the energy cascades downward to the smaller scales of motion. Experiments, however, have indicated that the anisotropy persists down to the small scales [5, 6]. Simple phenomenological considerations, as suggested by Lumley [7], indicate that the effect of anisotropy to a homogeneous turbulent field should lead to a $k^{-7/3}$ spectrum for the anisotropic part in addition to the isotropic $k^{-5/3}$ Kolmogorov spectrum.

The simplest anisotropic case is that of homogeneous shear flow with a uniform mean velocity gradient S in one direction, namely, $U_1 = Sx_2, U_2 = U_3 = 0$, where U_i are the components of the mean velocity field. It was Leslie [8] who first considered this case on the basis of Kraichnan's closure formulations [9]. Leslie's model is simple and intuitive which assumes an equivalent problem where an isotropic 'background' field is assumed to pre-exist which is assumed to be acted upon by the uniform shear. Thus the shear introduces a perturbation to the isotropic background, the latter having been treated by Kraichnan's direct interaction and test-field pictures [10, 11]. However, as has been pointed out recently [12, 13], Leslie's velocity correlation tensor does not satisfy the realizability conditions, namely, symmetry in the tensorial indices and solenoidality. It may be noted that it is much of a surprise if these realizability conditions are violated, because they must be satisfied by virtue of the definition of the correlation tensor in terms of the velocity field. Thus it is straightforward to guess that Leslie's calculations were not rigorous enough to maintain these realizability conditions. The simplicity and the intuitive appeal of Leslie's model tempts us to consider it more carefully.

In this paper, we offer a rigorous reformulation of Leslie's model leading to a general form for the velocity correlation tensor that satisfies realizability conditions of symmetry in its tensorial indices and the condition of solenoidality arising from the incompressibility condition together with possessing reflection invariance. The anisotropic part of the correlation tensor involves two non-dimensional constants—one arising from the pure strain term and the other from the term that induces distortion in the wave vector space; the rapid pressure term does not contribute. The estimates for these non-dimensional constants yield encouraging results within this framework.

2. Leslie's model

We shall begin with assuming a general form for the mean velocity: $U_i = S_{ij}x_j$, where the strain S_{ij} is uniform in space and constant in time, and $S_{ii} = 0$ due to incompressibility. On Reynolds decomposition [14–16] of the full velocity field, $v_i = U_i + u_i$, the Fourier transformed dynamics of the fluctuating part u_i is portrayed as

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) + \frac{\mathrm{i}}{2} P_{ijl}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p}, t) u_l(\mathbf{q}, t) = f_i(\mathbf{k}, t) + \lambda \hat{N}_{ij}(\mathbf{k}) u_j(\mathbf{k}, t), \tag{1}$$

with the operator

$$\hat{N}_{ij}(\mathbf{k}) = -S_{ij} + 2\frac{k_i k_l}{\mathbf{k}^2} S_{lj} + \delta_{ij} k_l S_{lm} \frac{\partial}{\partial k_m} \tag{2}$$

where the first term indicates pure strain, the second term is the rapid pressure term, whereas the last term signifies a distortion in the wave vector space [8, 14–16]. In equation (1), ν is the kinematic viscosity, $P_{ijl}(\mathbf{k}) = k_j P_{il}(\mathbf{k}) + k_l P_{ij}(\mathbf{k})$ with $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / \mathbf{k}^2$, the summation sign represents integrations on $\mathbf{p}$ and $\mathbf{q}$ with the triangle condition $\mathbf{p} + \mathbf{q} = \mathbf{k}$, and, following Leslie [8], λ (= 1) is inserted for use as an expansion parameter.

Leslie introduced the forcing term $f_i(\mathbf{k}, t)$ in equation (1) in order to maintain a steady state for the isotropic 'background' field $u_i^{(0)}(\mathbf{k}, t)$, whereas the λ-term is treated as a perturbation about this isotropic background. Thus, using the perturbation expansion

$$u_i(\mathbf{k}, t) = u_i^{(0)}(\mathbf{k}, t) + \lambda u_i^{(1)}(\mathbf{k}, t) + \cdots \tag{3}$$

and equating equal powers of λ leads to

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i^{(0)}(\mathbf{k}, t) + \frac{\mathrm{i}}{2} P_{ijl}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j^{(0)}(\mathbf{p}, t) u_l^{(0)}(\mathbf{q}, t) = f_i(\mathbf{k}, t) \tag{4}$$

and

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i^{(1)}(\mathbf{k}, t) + \mathrm{i} P_{ijl}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j^{(0)}(\mathbf{p}, t) u_l^{(1)}(\mathbf{q}, t) = \hat{N}_{ij}(\mathbf{k}) u_j^{(0)}(\mathbf{k}, t) \tag{5}$$

up to the first order.

Leslie treated the equation for the isotropic background field, equation (4), in the direct-interaction picture of Kraichnan [9], introducing a response tensor defined by the functional derivative $G_{ij}^{(0)}(\mathbf{k}; t, t') = \delta u_i^{(0)}(\mathbf{k}, t)/\delta f_j(\mathbf{k}, t')$, which is non-zero only for retarded effects $(t \geqslant t')$, leading to

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \delta u_i^{(0)}(\mathbf{k}, t) + \mathrm{i} P_{ijl}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j^{(0)}(\mathbf{p}, t) \delta u_l^{(0)}(\mathbf{q}, t) = \delta f_i(\mathbf{k}, t) \tag{6}$$

and

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) G_{im}^{(0)}(\mathbf{k}; t, t') + \mathrm{i} P_{ijl}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j^{(0)}(\mathbf{p}, t) G_{lm}^{(0)}(\mathbf{q}; t, t') = P_{im}(\mathbf{k}) \delta(t - t'), \tag{7}$$

with $t \geqslant t'$. It can be seen that

$$\delta u_i^{(0)}(\mathbf{k}, t) = \int_{-\infty}^{t} \mathrm{d}t' G_{ij}^{(0)}(\mathbf{k}; t, t') \delta f_j(\mathbf{k}, t'). \tag{8}$$

Thus, the first-order perturbation is obtained from equation (5) as

$$u_i^{(1)}(\mathbf{k}, t) = \int_{-\infty}^{t} \mathrm{d}t' G_{ij}^{(0)}(\mathbf{k}; t, t') \hat{N}_{jm}(\mathbf{k}) u_m^{(0)}(\mathbf{k}, t'). \tag{9}$$

Note that the integrand is in terms of the isotropic background field and the imposed anisotropy.

Based on the above equations, Leslie [8] found a form of the velocity correlation tensor (defined in equation (10)). It is easy to see that the correlation tensor must possess the properties of tensorial symmetry with respect to interchange of indices and the condition of solenoidality due to the incompressibility condition. However, Leslie's calculations were not rigorous enough to maintain these properties of the velocity correlation.

3. Rigorous reformulation of Leslie's model

As we have just noted, Leslie's calculations for the velocity correlation tensor were not rigorous enough to maintain the realizability conditions of tensorial symmetry and solenoidality. Thus, in treating the correlation tensors, we deviate from Leslie, and take a rigorous approach. The equal-time velocity correlation tensor is defined by

$$\langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', t)\rangle = Q_{ij}(\mathbf{k}; t, t) \delta^3(\mathbf{k} + \mathbf{k}') \tag{10}$$

where the angular brackets denote an ensemble average and $\delta^3(\mathbf{k} + \mathbf{k}')$ is the Dirac delta function in three dimensions. Using the perturbation expansion defined in equation (3), the correlation tensor can be expanded perturbatively about the isotropic background as $Q_{ij} = Q_{ij}^{(0)} + \lambda Q_{ij}^{(1)} + \cdots$, with

$$\left\langle u_i^{(0)}(\mathbf{k}, t) u_j^{(0)}(\mathbf{k}', t)\right\rangle = Q_{ij}^{(0)}(\mathbf{k}; t, t) \delta^3(\mathbf{k} + \mathbf{k}') \tag{11}$$

as the isotropic part, and the anisotropic part is given by

$$\left\langle u_i^{(1)}(\mathbf{k}, t) u_j^{(0)}(\mathbf{k}', t)\right\rangle + \left\langle u_i^{(0)}(\mathbf{k}, t) u_j^{(1)}(\mathbf{k}', t)\right\rangle = Q_{ij}^{(1)}(\mathbf{k}; t, t) \delta^3(\mathbf{k} + \mathbf{k}') \tag{12}$$

in the first order of the perturbation. Now we treat the first term in this equation by using equation (9), giving

$$\langle u_i^{(1)}(\mathbf{k},t)u_j^{(0)}(\mathbf{k}',t)\rangle = \int_{-\infty}^{t} \mathrm{d}t'\, G_{ia}^{(0)}(\mathbf{k};t,t')\langle u_j^{(0)}(\mathbf{k}',t)\hat{N}_{ab}(\mathbf{k})u_b^{(0)}(\mathbf{k},t')\rangle \quad (13)$$

whereas the second term can be obtained from equation (13) by the simultaneous interchanges $(i \leftrightarrow j)$ and $(\mathbf{k} \leftrightarrow \mathbf{k}')$, denoted henceforth by $[i, \mathbf{k} \leftrightarrow j, \mathbf{k}']$.

On using equation (2) in equation (13), it can be easily seen that the right-hand side of equation (12) involves three terms, which we will denote below by $Q_{ij}^{(1,1)}$, $Q_{ij}^{(1,2)}$ and $Q_{ijm}^{(1,3)}$ (excluding the Dirac delta functions), originating from the first, second, and third terms of equation (2), respectively. The pure shear term (first term of equation (2)) gives rise to

$$Q_{ij}^{(1,1)}(\mathbf{k},\mathbf{k}';t,t) = -S_{ab}\int_{-\infty}^{t} \mathrm{d}t'\, G_{ia}^{(0)}(\mathbf{k};t,t')Q_{jb}^{(0)}(\mathbf{k}';t,t') + [i,\mathbf{k} \leftrightarrow j,\mathbf{k}'] \quad (14)$$

with a note that the integrand contains two-time velocity correlation of the isotropic background field.

The property of isotropy of the background field demands that isotropic parts of the response and correlation tensors are expressible as $X_{ij}^{(0)}(\mathbf{k};t,t') = X^{(0)}(k;t,t')P_{ij}(\mathbf{k})$, where the symbol X stands for either G or Q. Thus the rapid pressure term (second term of equation (2)) yields no contribution, namely, $Q_{ij}^{(1,2)} = 0$.

Now we shall consider the contribution coming from the third term of equation (2), namely, $Q_{ijm}^{(1,3)}$, which gives the contribution due to the deformation in the wave vector space due to the imposed mean strain. This term involves the derivative with respect to the wave vector components. Transforming back to the configuration space by means of the inverse Fourier transform

$$u_i^{(0)}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x\, u_i^{(0)}(\mathbf{x},t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \quad (15)$$

and using

$$Q_{ij}^{(0)}(\mathbf{x}-\mathbf{x}';t,t') = \langle u_i^{(0)}(\mathbf{x},t)u_j^{(0)}(\mathbf{x}',t')\rangle = \int \mathrm{d}^3k\, Q_{ij}^{(0)}(\mathbf{k};t,t')\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \quad (16)$$

we obtain the result

$$\left\langle u_i^{(0)}(\mathbf{k}',t)\frac{\partial u_j^{(0)}(\mathbf{k},t')}{\partial k_m}\right\rangle = Q_{ij}^{(0)}(\mathbf{k}';t,t')\frac{\partial\delta^3(\mathbf{k}+\mathbf{k}')}{\partial(k_m+k_m')}. \quad (17)$$

Using this result, we obtain from equation (9)

$$Q_{ij}^{(1)}(\mathbf{k};t,t)\delta^3(\mathbf{k}+\mathbf{k}') = Q_{ij}^{(1,1)}(\mathbf{k},\mathbf{k}';t,t)\delta^3(\mathbf{k}+\mathbf{k}') + Q_{ijm}^{(1,3)}(\mathbf{k},\mathbf{k}';t,t)\frac{\partial\delta^3(\mathbf{k}+\mathbf{k}')}{\partial(k_m+k_m')}, \quad (18)$$

with

$$Q_{ijm}^{(1,3)}(\mathbf{k},\mathbf{k}';t,t) = \delta_{ab}S_{lm}\int_{-\infty}^{t} \mathrm{d}t'\, k_l G_{ia}^{(0)}(\mathbf{k};t,t')Q_{jb}^{(0)}(\mathbf{k}';t,t') + [i,\mathbf{k} \leftrightarrow j,\mathbf{k}']. \quad (19)$$

Now integrating both sides of equation (18) over $\mathbf{k}'$, keeping $\mathbf{k}$ fixed, we arrive at

$$Q_{ij}^{(1)}(\mathbf{k};t,t) = Q_{ij}^{(1,1)}(\mathbf{k},-\mathbf{k};t,t) - \left[\frac{\partial}{\partial k_m'}Q_{ijm}^{(1,3)}(\mathbf{k},\mathbf{k}';t,t)\right]_{\mathbf{k}'=-\mathbf{k}} \quad (20)$$

where the last term results from integrating the last term of equation (18) by parts, contribution from the surface term being zero. The right-hand terms of equation (20) can be readily evaluated using the expressions in equations (14) and (19) and the property of isotropy of the response and correlation tensors associated with the background field, leading to

$$Q_{ij}^{(1)}(\mathbf{k};t,t) = S_{ab}[P_{ia}(\mathbf{k})P_{jb}(\mathbf{k}) + P_{ja}(\mathbf{k})P_{ib}(\mathbf{k})]R^{(1,1)}(k,t) + S_{lm}\frac{k_l k_m}{\mathbf{k}^2}P_{ij}(\mathbf{k})R^{(1,3)}(k,t) \tag{21}$$

where the terms involving the scalars $R^{(1,1)}$ and $R^{(1,3)}$ come from the terms involving $Q_{ij}^{(1,1)}$ and $Q_{ijm}^{(1,3)}$ of equation (20), respectively, and

$$R^{(1,1)}(k,t) = -\int_{-\infty}^{t} \mathrm{d}t' G^{(0)}(k;t,t')Q^{(0)}(k;t,t') \tag{22}$$

$$R^{(1,3)}(k,t) = \int_{-\infty}^{t} \mathrm{d}t' k \left[G^{(0)}(k;t,t')\frac{\partial Q^{(0)}(k;t,t')}{\partial k} - Q^{(0)}(k;t,t')\frac{\partial G^{(0)}(k;t,t')}{\partial k}\right]. \tag{23}$$

The anisotropic correlation tensor $Q_{ij}^{(1)}(\mathbf{k};t,t)$, obtained as equation (21), clearly displays that it is symmetric with respect to the interchange of tensorial indices, namely $Q_{ij}^{(1)}(\mathbf{k};t,t) = Q_{ji}^{(1)}(\mathbf{k};t,t)$, and it also satisfies the condition of solenoidality, namely $k_i Q_{ij}^{(1)}(\mathbf{k};t,t) = 0$.

We further observe that the specific form of the anisotropic correlation tensor $Q_{ij}^{(1)}(\mathbf{k};t,t)$, given by equation (21), resulted from our rigorous calculations. It is interesting to note that this form is congruent with the form first postulated in [12] where a direct numerical simulation (DNS) was performed.

4. Results

We take an exponential decay of the response, as also assumed by Leslie, namely

$$\begin{aligned} G^{(0)}(k;t,t') &= \mathrm{e}^{-\eta(k)(t-t')} && \text{for } t \geqslant t' \\ &= 0 && \text{otherwise.} \end{aligned} \tag{24}$$

Further, like Leslie, we shall take Kraichnan's test-field closure [10, 11] so that

$$Q^{(0)}(k;t,t') = Q^{(0)}(k;t,t)G^{(0)}(k;t,t'). \tag{25}$$

We shall also follow the prescription of switching to three η's in the denominators as suggested by Leslie [8].

Using these assumptions, and carrying out the integrations in equations (22) and (23), we obtain

$$R^{(1,1)}(k) = -\frac{1}{3}\frac{q^{(0)}(k)}{\eta(k)} \qquad \text{and} \qquad R^{(1,3)}(k) = \frac{1}{3}\frac{k}{\eta(k)}\frac{\partial q^{(0)}(k)}{\partial k}, \tag{26}$$

where $q^{(0)}(k) = Q^{(0)}(k;t,t)$. The Kolmogorov scaling [1–3] suggests that

$$q^{(0)}(k) = \frac{C}{4\pi}\bar{\varepsilon}^{2/3}k^{-11/3}, \tag{27}$$

where C is the Kolmogorov constant appearing in the relation for energy spectrum, $E(k) = C\bar{\varepsilon}^{2/3}k^{-5/3}$, and $\bar{\varepsilon}$ is the energy dissipation rate. Kraichnan [11] defined another constant μ given by $\eta(k) = \mu\sqrt{C}\bar{\varepsilon}^{1/3}k^{2/3}$. Using these scaling laws, we get

$$R^{(1,1)}(k) = A\bar{\varepsilon}^{1/3}k^{-13/3} \qquad \text{and} \qquad R^{(1,3)}(k) = B\bar{\varepsilon}^{1/3}k^{-13/3}, \tag{28}$$

where

$$A = -\frac{1}{12\pi}\frac{\sqrt{C}}{\mu} \qquad \text{and} \qquad B = -\frac{11}{36\pi}\frac{\sqrt{C}}{\mu}. \tag{29}$$

Thus we obtain the ratio

$$\frac{B}{A} = \frac{11}{3}. \tag{30}$$

This ratio may be compared with those obtained by DNS, namely ~2.5 in [12], and by experiment, namely ≈2.65 in [17].

We can also obtain estimates for the non-dimensional numbers A and B. The ratio $C/\mu^{2/3}$ was evaluated by Kraichnan [11] numerically using the energy transport equation of the DIA, yielding $C = 3.022\mu^{2/3}$. Using this result, we can obtain the non-dimensional constants in terms of the Kolmogorov constant:

$$A = -\frac{1}{12\pi}\frac{(3.022)^{3/2}}{C} \qquad \text{and} \qquad B = -\frac{11}{36\pi}\frac{(3.022)^{3/2}}{C} \tag{31}$$

so that, using the test field value $C = 1.4$ [11], obtained by Kraichnan via numerical integration of the test-field closure equations, yields $A = -0.10$ and $B = -0.37$.

On the other hand, using the analytical result $C = 1.64$ [18], obtained anlytically by means of perturbative evaluation (double expansion), yields $A = -0.09$ and $B = -0.31$.

5. Discussions

We carried out a rigorous reformulation of Leslie's model yielding a general form for the velocity correlation tensor. This correlation is found to possess the realizability conditions of symmetry with respect to interchange of indices and the property of solenoidality coming from incompressibility. However, Leslie's original formulation was unable to maintain these properties. It is expected that these realizability conditions must be satisfied by virtue of the definition of the correlation tensor. Therefore, it comes as a surprise when Leslie's model violated these conditions. To solve this puzzle, we have performed a rigorous reformulation of the Leslie's model and shown that Leslie's model indeed reproduces both the above realizability conditions for the correlation tensor. Moreover, the present formulation reproduces the specific tensorial form which has been postulated recently [12] where a DNS was performed.

The quantitative estimates within this reformulated Leslie's model are also impressive. The estimates for the non-dimensional constants, namely $A = -0.10$ and $B = -0.37$, obtained in our calculations presented herein are encouraging when we consider the level of approximations involved in the calculations. These values are not far from the DNS values $A = -0.16 \pm 0.03$ and $B = -0.40 \pm 0.06$ [12] and the experimental values $A \approx -0.17$ and $B \approx -0.45$ [17]. The values $A = -0.09$ and $B = -0.31$, based on the perturbative evaluation of the test-field closure equations, although slightly underestimate the numbers, are acceptable. It may be noted that analytical calculations based on the Lagrangian renormalized approximation (LRA) [13] yields $A = -0.12 \pm 0.002$ and $B = -0.009 \pm 0.014$; the value of B is rather unsatisfactory when compared with the DNS and experimental values. In comparison, the value of B predicted by Leslie's model is in better agreement with DNS and experiment. These results clearly indicate the power of the underlying assumptions in Leslie's model and encourage us to consider Leslie's model seriously. As future work, it would be worthwhile to carry out a renormalization group analysis of the Leslie's model in the context of earlier work [19–21]. It may also be noted that an approach similar to that presented here can be applied to other physical problems such as zonal flow generation in an anisotropic turbulence regime [22].

We would also like to discuss about the nature of approximation used in the present calculation. The perturbation expansion in the parameter λ seems to suggest that the results are applicable only for weak anisotropy cases. However, when we consider the next higher order terms in the perturbation expansion, it can be seen that they lead to additional terms for the anisotropic correlation which scale as k^{-5}. This order of contribution is irrelevant as compared to the present order of contribution ($k^{-13/3}$) in the high wavenumber regime (cf equations (28)). Correspondingly, the $k^{-13/3}$ contributions would dominate over the k^{-5} contributions even for higher shear cases.

Further, the fact that the total energy should be positive definite can be regarded as an additional realizability condition in addition to the previously quoted ones, namely solenoidality and symmetry. The total energy W can be calculated by taking the trace of the correlation tensor and integrating over the k-space, namely $W = \frac{1}{2}\int Q_{ii}(\mathbf{k})\,\mathrm{d}^3k$. Noting that $Q^{(1)}_{ii}(\mathbf{k}) = 2[R^{(1,3)}(k) - R^{(1,1)}(k)]S_{lm}k_lk_m/\mathbf{k}^2$, we see that the integral $\int Q^{(1)}_{ii}(\mathbf{k})\,\mathrm{d}^3k$ vanishes because the angular integration generates a Kronecker delta, namely δ_{lm}. Thus the anisotropic part of the correlation tensor contributes nothing to the total energy W. Consequently, it is only the isotropic part, namely $Q^{(0)}_{ij}(\mathbf{k}) = q^{(0)}(k)P_{ij}(\mathbf{k})$, that contributes to the total energy W. Calculation yields a positive quantity, viz, $W \sim \bar{\varepsilon}^{2/3}L^{2/3}$, where L is the scale associated with the largest eddies in the flow region.

We also note that the form of the anisotropic part of correlation tensor as obtained in equation (21) (and hence the realizability properties of its solenoidality and symmetry in the indices) is an automatic consequence of the dynamics given by Leslie's model. It contains the scaling functions $R^{(1,1)}(k)$ and $R^{(1,3)}(k)$ which are given by equations (28) and we have demonstrated that they are calculable. Quite interestingly, a recent work has shown that a general form of the correlation tensor can be obtained by imposing the conditions of solenoidality and symmetry and using geometric properties related with tensors [23].

We finally note that the phenomenon of intermittency in homogeneous anisotropic turbulence has been analysed by decomposing the correlation function (and the structure functions) in the irreducible representations of the $SO(3)$ rotation group [24]. This analysis gives the (anomalous) scaling exponents belonging to different sectors of the symmetry group. Particularly, a simplified model (linear pressure model) yields the exponents $\frac{11}{3}$, 4.25, 5.02, 7.05, ..., whereas a closure approximation yields $\frac{11}{3}$, 4.36, 4.99, 6.98, ... in the sectors $\ell = 0, 2, 4, 6, \ldots$, respectively. The deviations of these values from $\frac{11}{3}, \frac{13}{3}, 5, \frac{17}{3}, \ldots$ demonstrates the role of intermittency in the anisotropic sectors $\ell = 2, 4, 6, \ldots$ within those model calculations. It can be guessed that the anomalous scaling exponents form an increasing spectrum thereby suggesting that the effect of anisotropy diminishes with decreasing scale.

Acknowledgments

The author gratefully acknowledges the hospitality at the Friedrich Alexander-Universität, Erlangen, Germany, and the support of the Deutscher Akademischer Austausch Dienst (DAAD). Further support of LSTM Erlangen is also acknowledged.

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